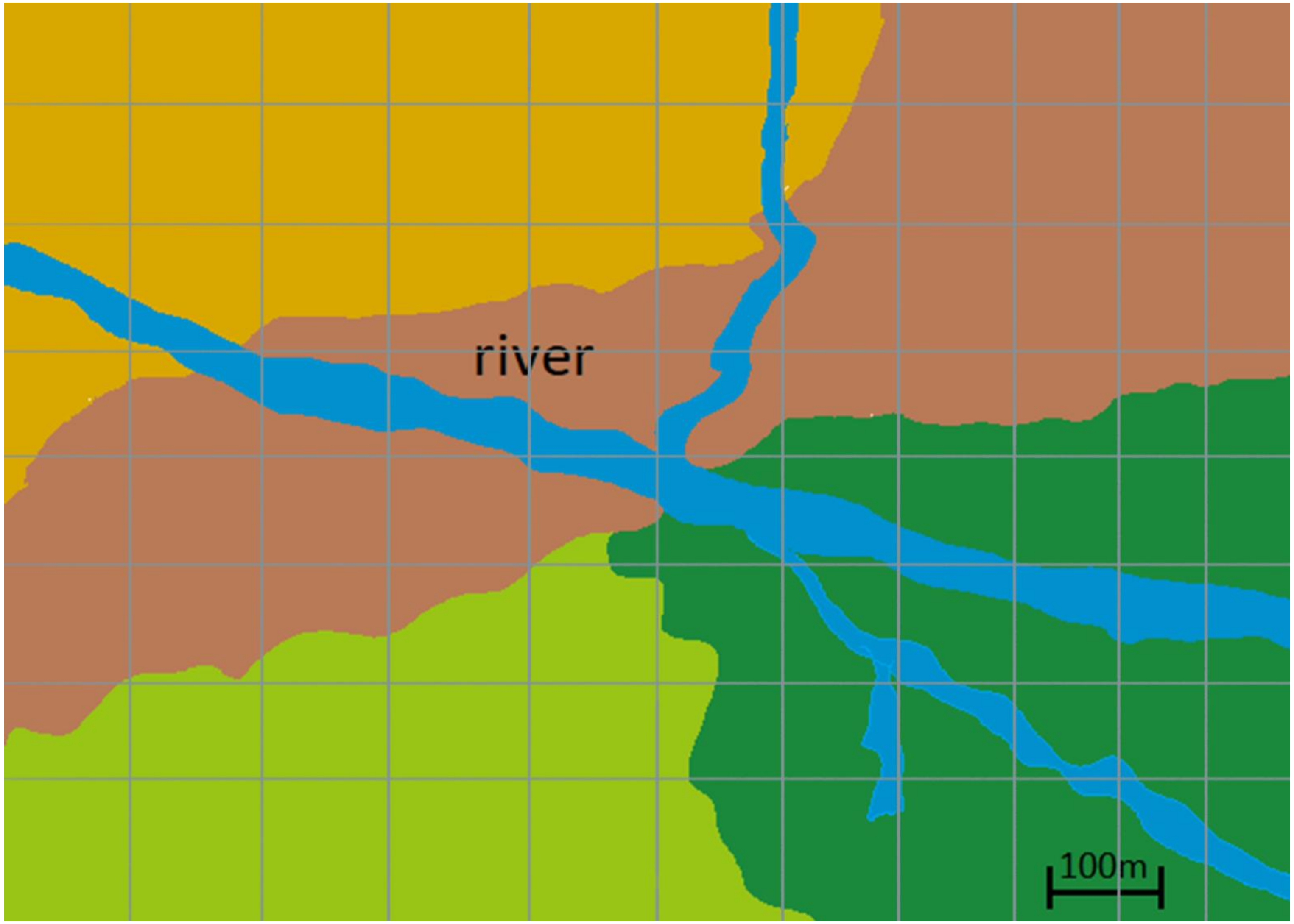
Hydraulic groundwater modeling

- Week 4

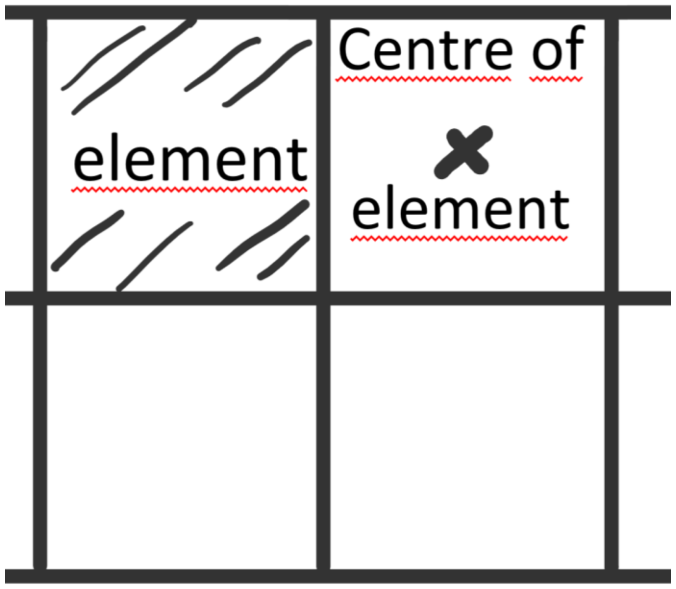
- Spatial discretization

Continuous world – discrete model

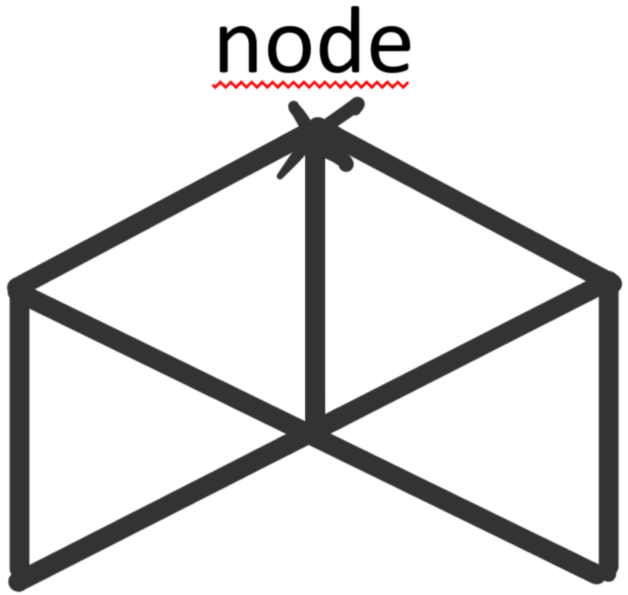


Types of elements

- Quadrilateral elements

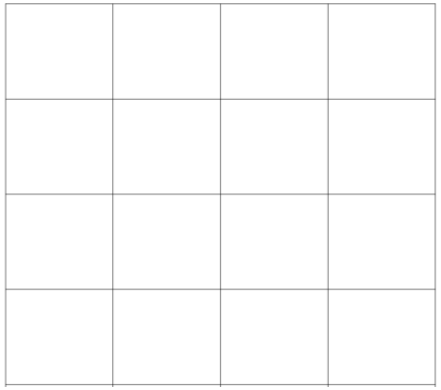


Triangular elements

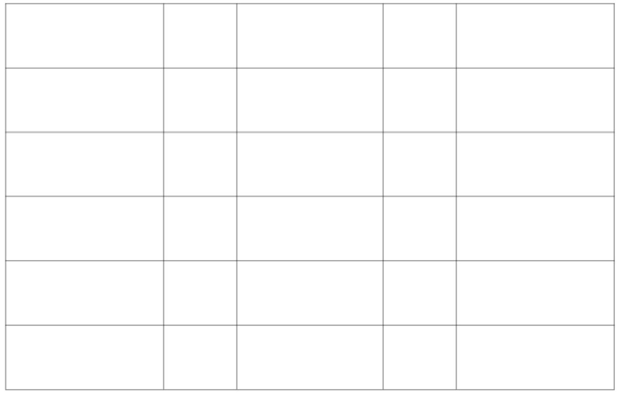


Structured meshes

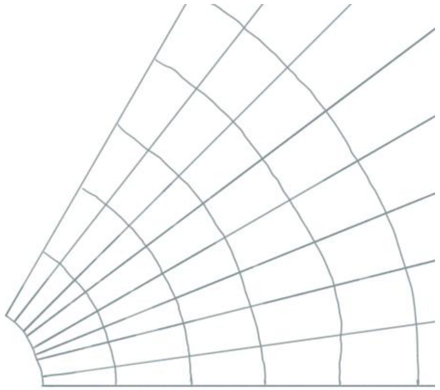
- Regular connectivity
- Memory efficient -> translates to array easily
- Good convergence
- Examples:
  - Cartesian grid



Rectilinear grid

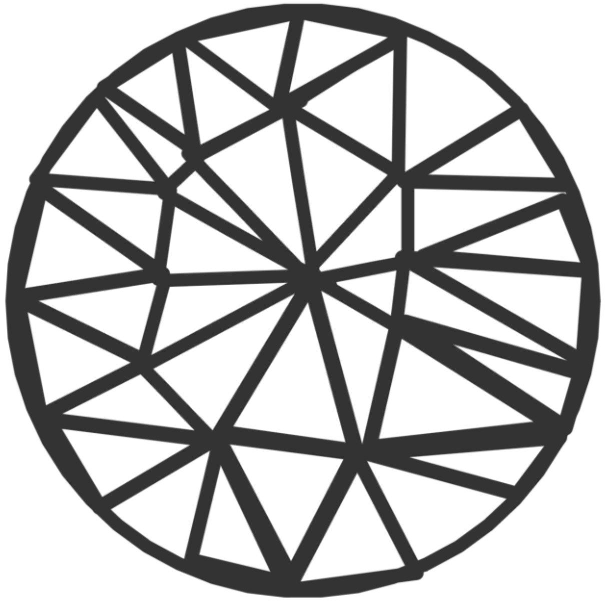


Curvlinear grid



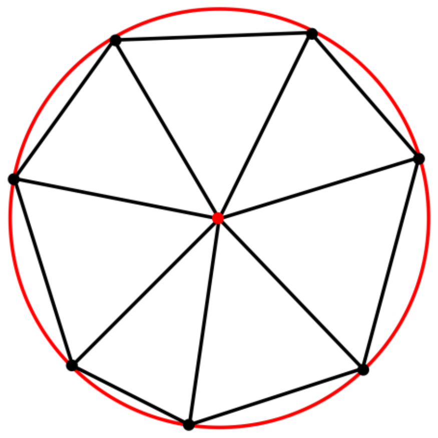
Unstructured meshes

- Irregular connectivity
- Requires more memory (explicit storage of neighbors)
- Typically employ triangles in 2D and tetrahedral in 3D
- Great for complex geometries
- Mesh can be refined, modified quite easily

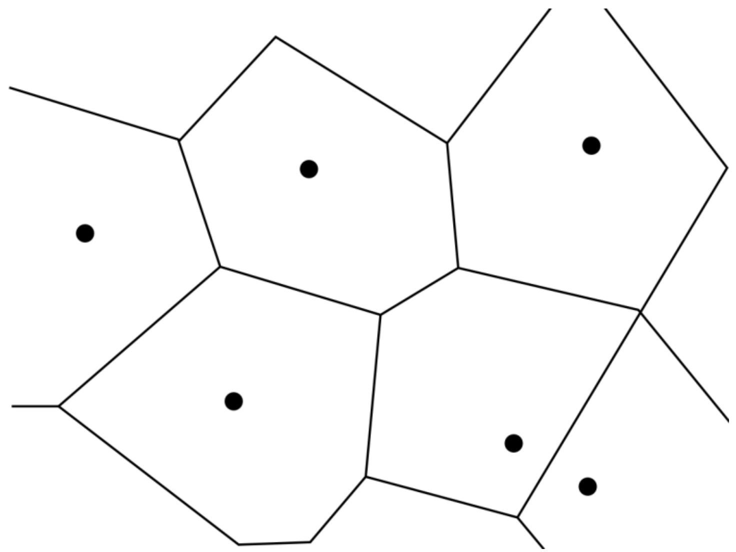


Generation of unstructured meshes

- Delauney triangulation
  - Conecting points to triangles, so that no other points are within circumscribed circle

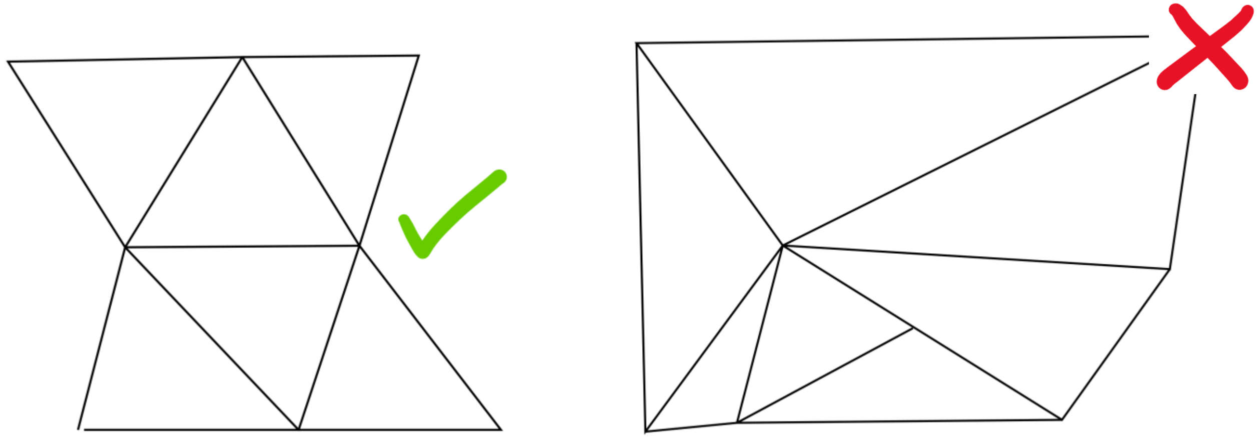


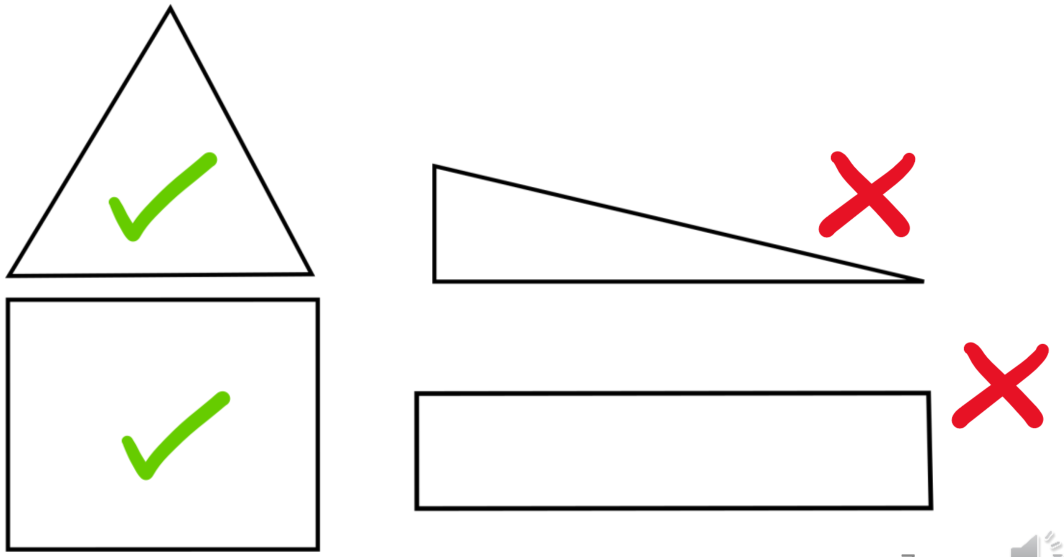
- Voronoi tessalation
  - Observation points are the centers and any point in the surrounding cell is closer to the center than to any other center.



Mesh quality

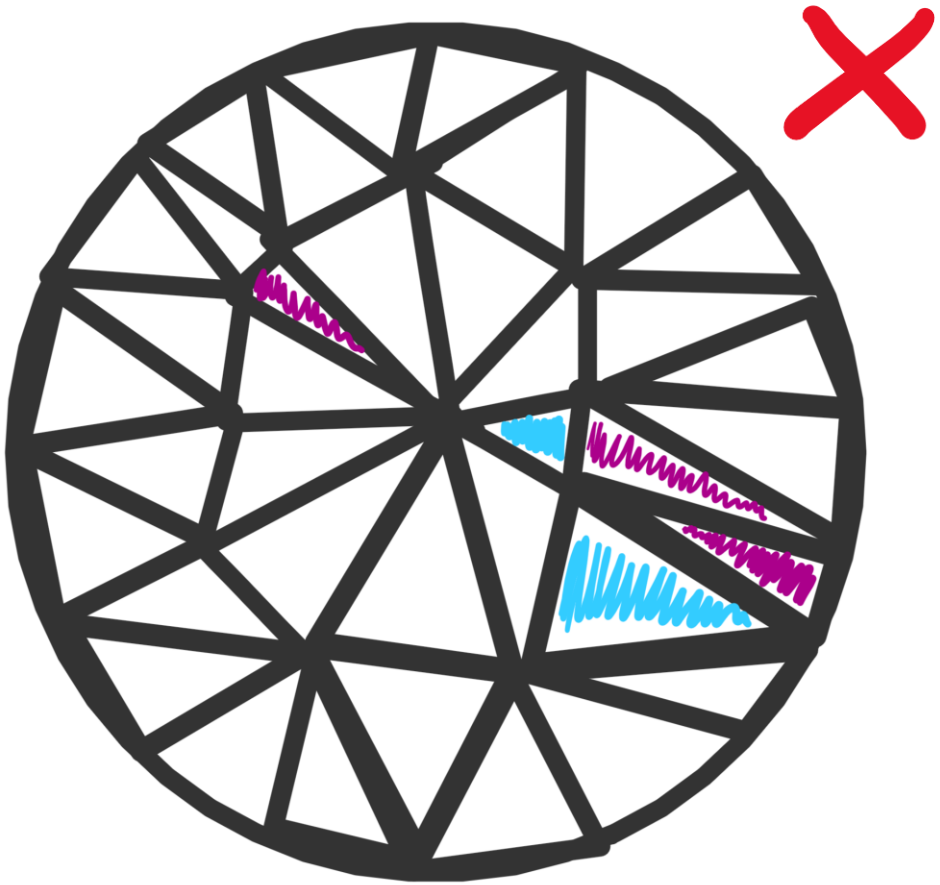
- **Goal:** Obtaining a more accurate solution more quickly
- A preferable element shape has:
  - a smooth change in neighboring cell sizes
  - a good aspect ratio (equal length of all borders)
  - no sharp angles





Example mesh: the good and the bad

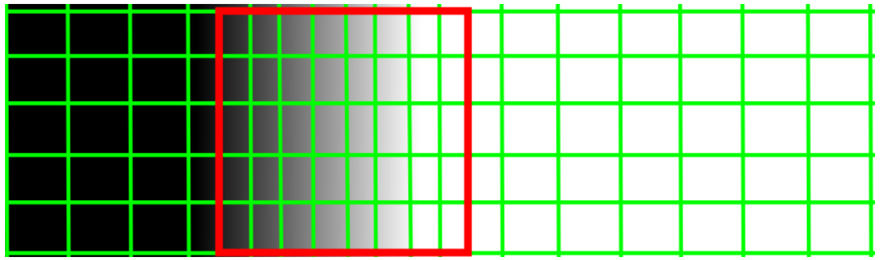
- Try to avoid:
  - large contrast in element size
  - Sharp angles
  - Aspect ratio much smaller than 1
  - Too large/too small element size

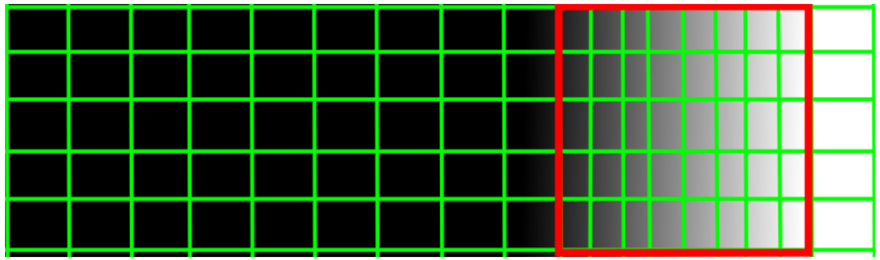


Mesh refinement

- More points/cells often related with higher accuracy
- BUT: More points = more computational costs

- mesh refinement:
  - Adding nodes in regions of high solution gradients
  - During runtime: *adaptive mesh refinement* (https://youtu.be/u-VV3euIsXo)



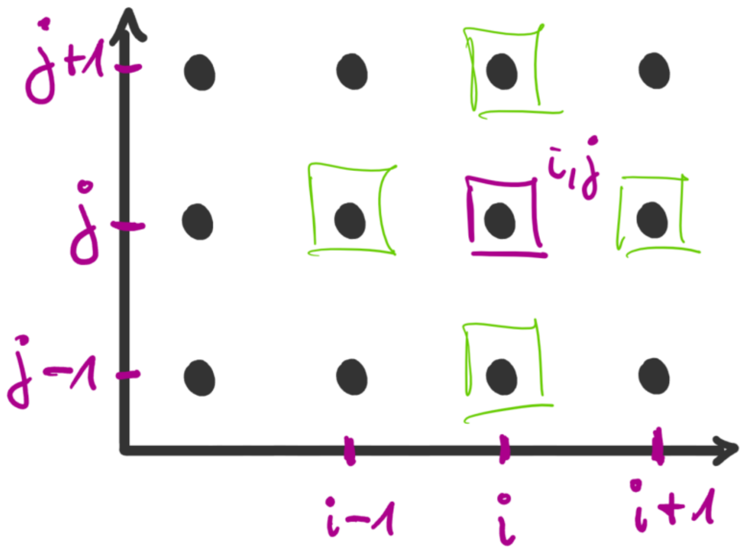


Physics on discrete points

- Finite differences:
  - The oldest, most popular, and conceptually simplest method
  - Uses structured meshes
  - Based on relationship between discrete points
- Finite elements:
  - Nowadays, probably the most common method
  - Uses unstructured meshes
  - Based on integration over elements
- Finite volumes:
  - Especially common in fluid dynamics
  - Uses structured and unstructured meshes
  - Based on integration over volumes and fluxes across surfaces

Finite differences

- Based on the Taylor series expansion of derivatived functions
- Depending on the involved grid points, the numerical schemes vary
- Higher order schemes involve more grid points
- Accuracy and stability vary with the used stencil

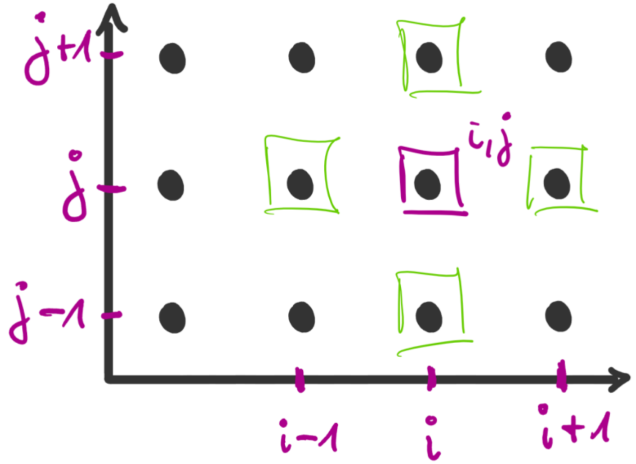


$$\frac{\partial f}{\partial x} = \frac{f_{i+1,j} - f_{i-1,j}}{2\,dx} + O(dx^2)$$

$$\frac{\partial f}{\partial y} = \frac{f_{i,j+1} - f_{i,j-y}}{2\,dy} + O(dy^2)$$

Finite differences II

- There are other stencils
  - Forward in space $$\frac{\partial f}{\partial x} = \frac{f_{i+1,j} - f_{i,j}}{dx} + O(dx^2)$$
  - Backward in space $$\frac{\partial f}{\partial x} = \frac{f_{i,j} - f_{i-1,j}}{dx} + O(dx^2)$$



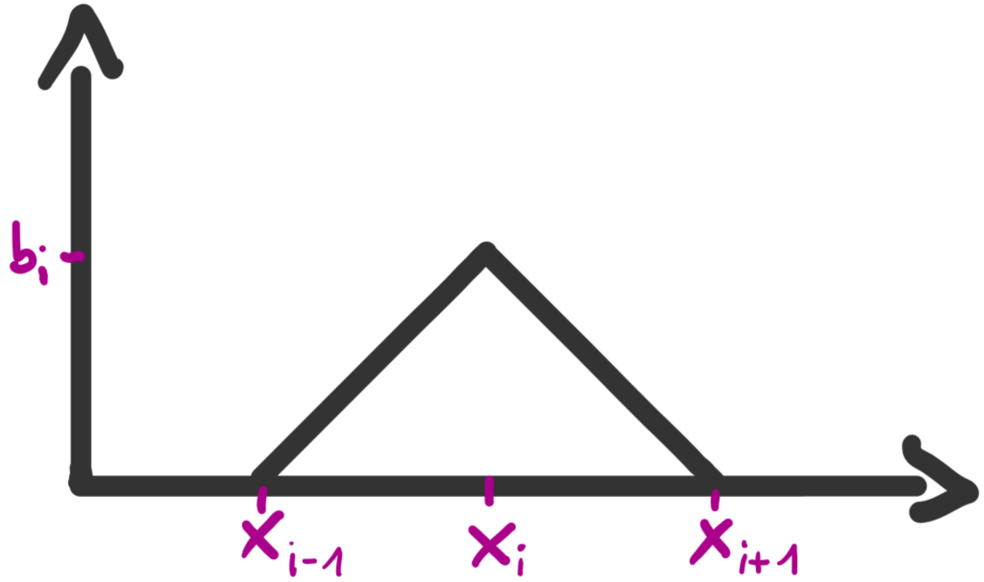
- There are higher order stencils
  - Centered in space $$\frac{\partial f}{\partial x} = \frac{-f_{i+2,j} + 8f_{i+1,j} - 8f_{i-1,j} + f_{i-2,j}}{12\, dx} + O(dx^2)$$
- There are stencils for higher order derivatives

Finite differences III

- Fast, flexible & easy (also for developers)
- Limitations on geometry (at least at first sight)
- A lot of extensions exist (markers-in-cell, staggered grid, etc…)
- There is a lot of literature available:
  - Press et al. (2007): Numerical Recipes, 3rd edition, Cambridge University Press
  - Gerya (2009): Introduction to Numerical Geodynamic Modeling, Cambridge University Press

Finite elements

- For the finite element method an **integral approach** is applied.
- Most common: the weighted residual (Galerkin) technique
- The Galerkin method:
  - Approximate value is a combination of the real value in node i and a shape function b(x, y)



$$f_{apparent}(\mathrm{x}, \mathrm{y}) = \sum_{i=1}^{N} f_i b_i$$

Finite elements

- Approximate a function with a set of basis functions
- Coefficients corresponding to the functional values at some nodes
- Solution obtained by solving a linear system of equations

- Can be applied to any kind of geometry as long as meshing is good
- Elements can be placed anywhere. Variable mesh density
- Nodes can be easily added/removed & elements can be deformed

Finite volumes

- Based on Gauss' theorem
  - Gauss' theorem $$\int_V \nabla \vec{q}\, dV = \oint_S \vec{q}\vec{n}\, dS$$
  - The divergence is transferred into a flux through a surface boundary
- Conservative: What leaves one cell, enters the next
- Requires the calculation of Voronoi cells
- For structured and unstructured meshes

Lessons learned

- Mesh quality & mesh refinement
- Different kind of meshes incl. their pros and cons
- Delauney triangulation and Voronoi cells
- Different approaches to discretize space (benefits/drawbacks)