Hydraulic groundwater modeling

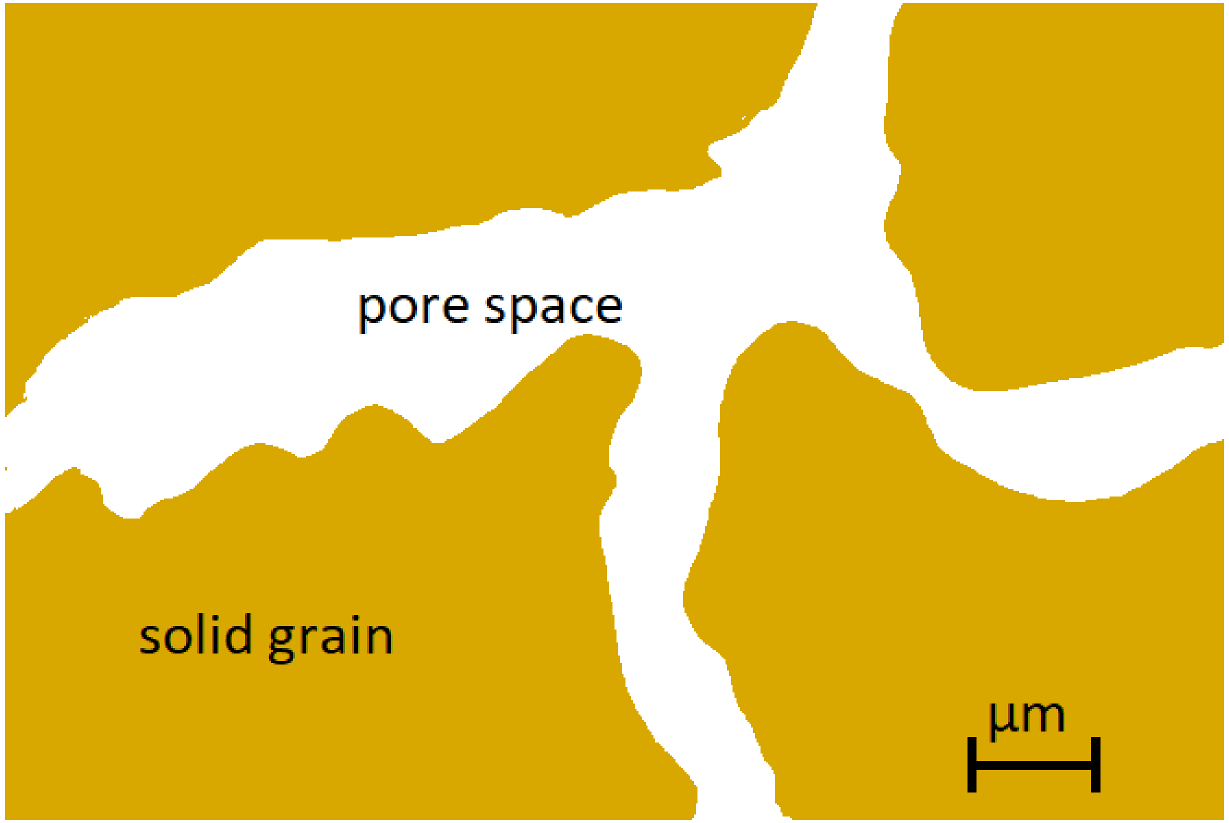
- Week 2
- Governing equations for groundwater flow and solute transport

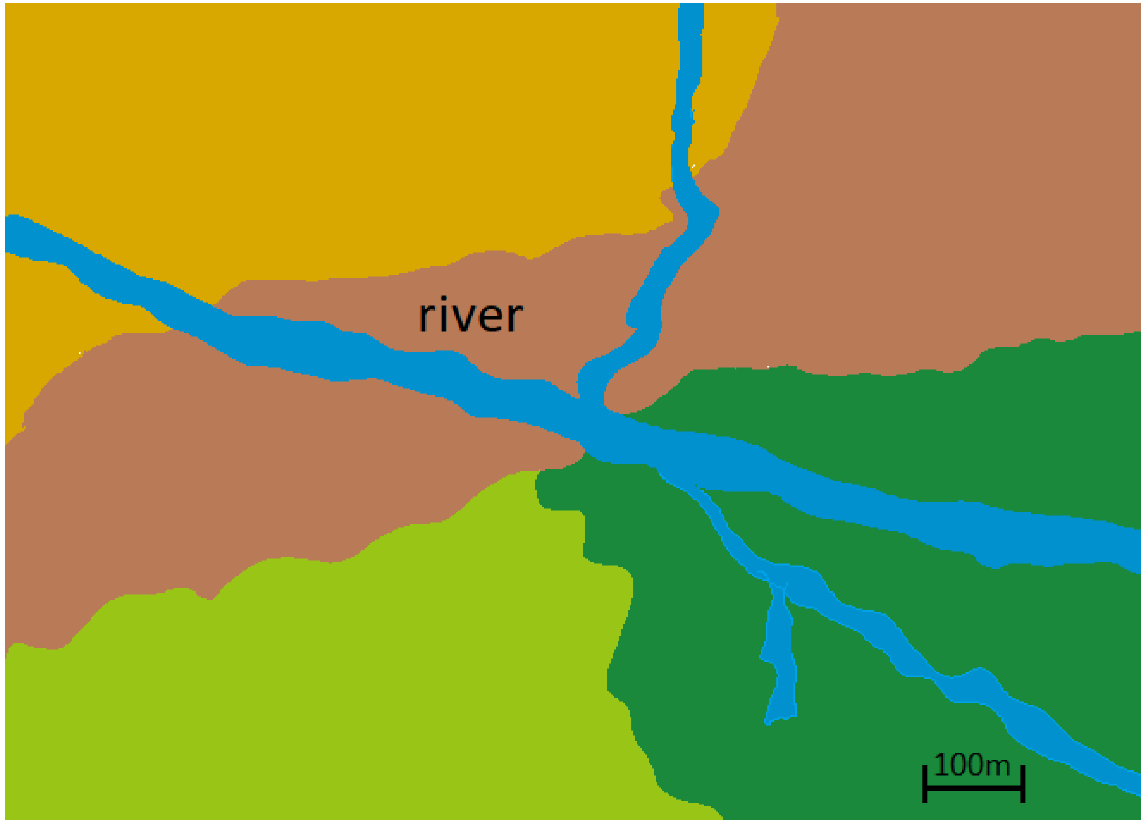
Continuum mechanics

- Considering any physical mass as a continuum in opposition to discrete particles.
- We are not looking at:
  - individual fluid or tracer particles
  - separate solid/soil grains
- Instead, we look at:
  - the soil/rock as a continuous porous medium described by density, porosity, permeability, etc.
  - the fluid and tracer as a continuous phase within the void space described by pressure, concentration, density, viscosity, etc.

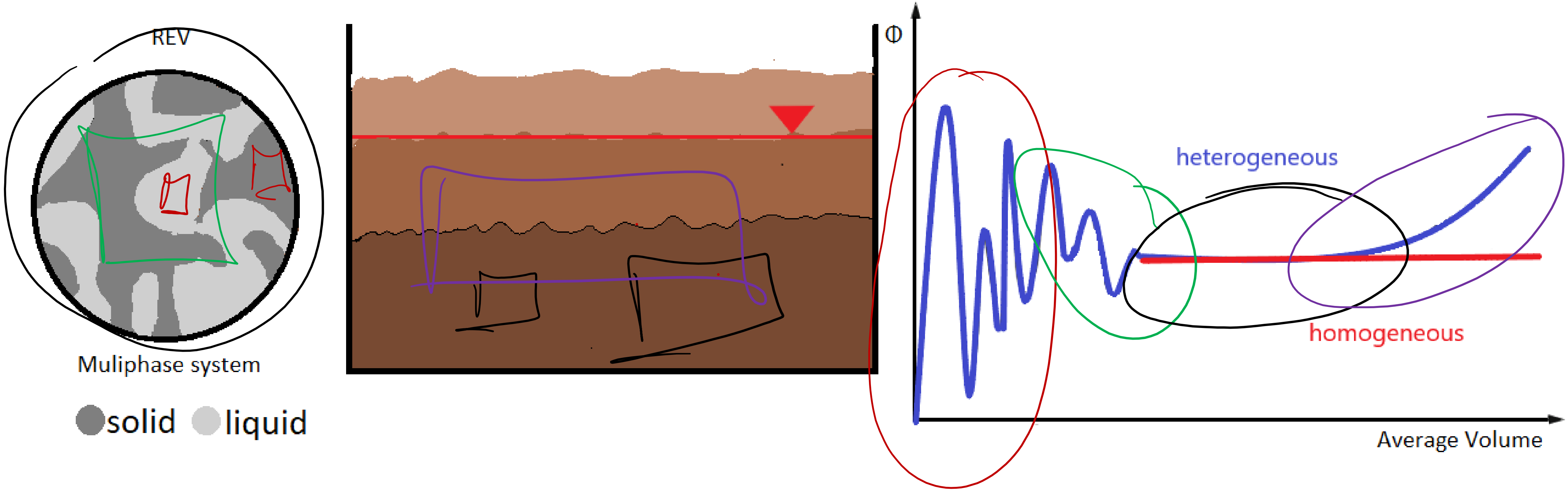
Spatial scale

The applied mathematical equations (and associated parameters) depend on the chosen scale





Representative elementary volume (REV)



The REV

- The REV is filled with solid and void space
- It is representative for its subdomain.
- Its properties are statistically meaningful.
- Fluctuations of spatially averaged values are negligible.

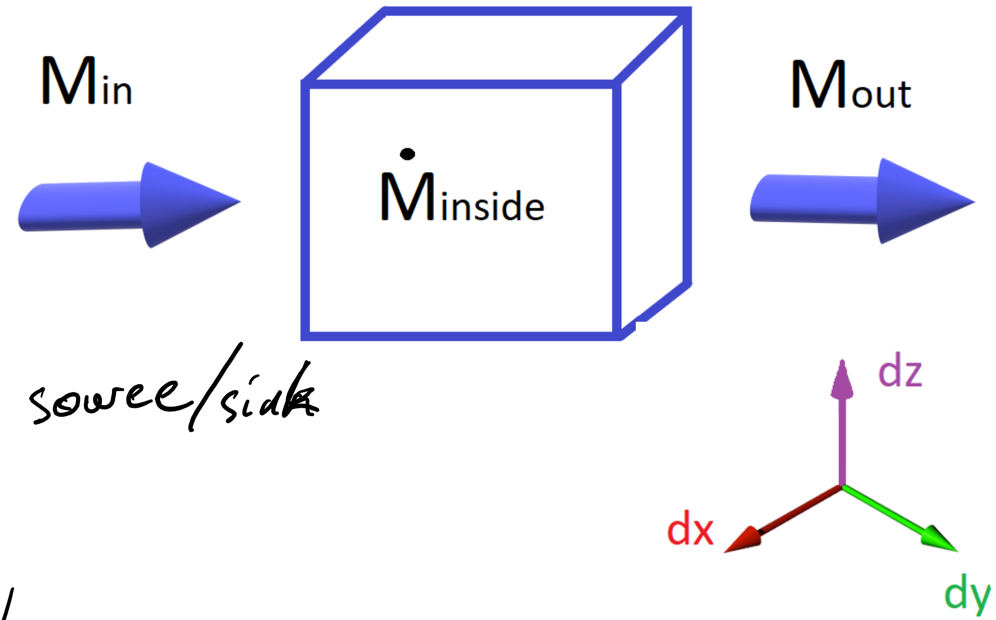
- → Resulting averaged quantities are independent of the REV size!

The conservation of mass

ϕ - porosity

ρ - density fluid

$$\underbrace{\boxed{\frac{d\phi\rho}{dt}}}_{\dot{M}_{inside}} = \overbrace{\underbrace{\boxed{-\rho\nabla\vec{q}}}_{\text{Darcy velocity}}}^{\substack{M_{out} - M_{in} \\ \text{fluxes}}} + \overset{\text{source/sink}}{Q}$$



Left hand side

- Chain rule: $$\frac{d\phi\rho}{dt} = \rho\frac{d\phi}{dt} + \phi\frac{d\rho}{dt}$$
- Dependencies $$\frac{d\phi(P,T,S)}{dt} = \underline{\boxed{\frac{\partial\phi}{\partial P}}\frac{\partial P}{\partial t}} + \cancel{\underline{\frac{\partial\phi}{\partial T}\frac{\partial T}{\partial t}}} + \cancel{\underline{\frac{\partial\phi}{\partial S}\frac{\partial S}{\partial t}}} \cancel{+ \dots}$$
- Compressibility $$c_\phi = \frac{1}{\phi}\frac{\partial\phi}{\partial P}$$

Right hand side

- Darcy's law $$\vec{q} = -\frac{\overset{\leftarrow \text{permeability}}{k}}{\underset{\nwarrow \text{viscosity}}{\eta}} \nabla P$$

- Final result

$$\rho\phi \underset{\substack{\uparrow \\ c_\phi + c_f}}{c_t} \frac{\partial P}{\partial t} = \rho \boxed{\nabla \left(\frac{k}{\eta} \nabla P \right)} + Q$$

Hydraulic head

- From pressure to hydraulic head $P = \rho \overset{\text{grav. acceleration}}{g} h$
- Intermediate step $$\phi c_t \rho g \frac{\partial h}{\partial t} = \nabla \left(\frac{k\rho g}{\eta} \nabla h \right) + Q$$
- Change in parameters
  - Hydraulic conductivity K (m/s)

specific storage S (1/m)

$$K = \frac{k\rho g}{\eta}$$

$$S_s = \rho g \phi c_t$$

$$\boxed{S_s \frac{\partial h}{\partial t} = \nabla \left(K \nabla h \right) + Q}$$

Horizontal groundwater flow

- An aquifer is usually wider than deep & we are not interested in variations with depth -> depth integration

z aquifer thickness

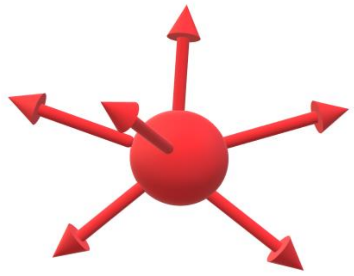
$$\boxed{\int_0^m} S_s \frac{\partial h(x,y,z,t)}{\partial t} dz = \int_0^m \left(\nabla \left(K \nabla h(x,y,z,t)\right) + Q\right) dz$$

- Change in parameters: Transmissivity and Storativity

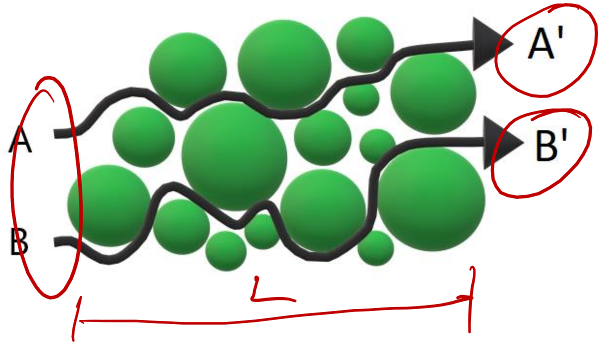
$$\boxed{S} \frac{\partial h(x,y,t)}{\partial t} = \nabla \left(\boxed{T} \nabla h(x,y,t)\right) + Q$$

Transport processes

- Diffusion $\vec{j}_{dif} = -D\nabla C$



- Dispersion $\vec{j}_{dis} = -\mathbf{D}_s\nabla C$



- Advection $\vec{j}_{adv} = \vec{q}C$

Retardation and Decay

- Adsorption/Retardation: reduces concentration & slows spreading

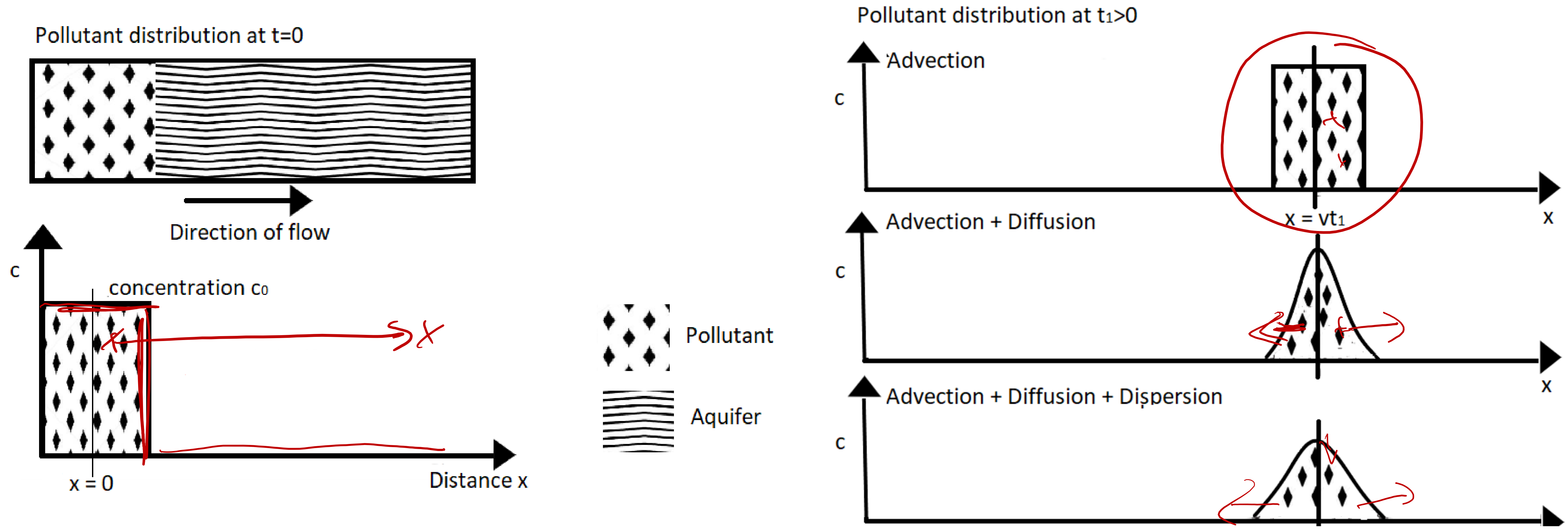
$$\frac{\partial C_{ads}}{\partial t} = -\frac{\rho_b}{\phi} K_d \frac{\partial C}{\partial t}$$

$$R_d = 1 + \frac{\rho_b}{\phi} K_d$$

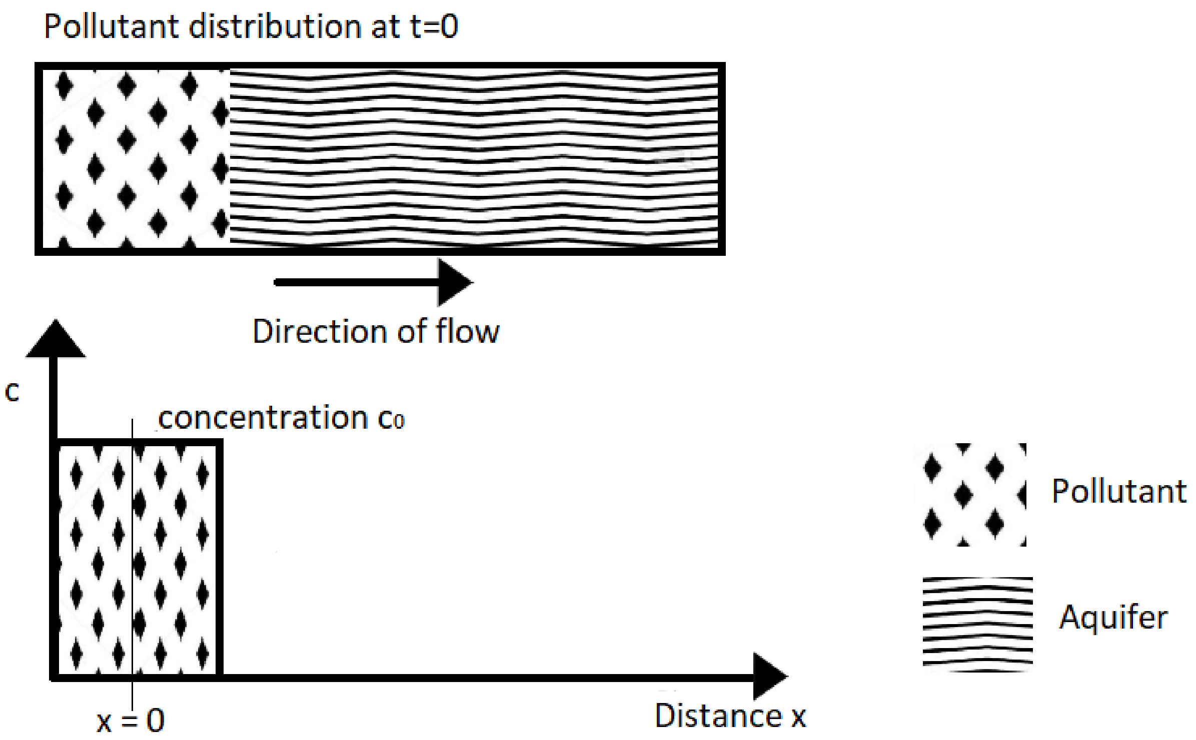
- Vanishing concentration through biological or chemical decay

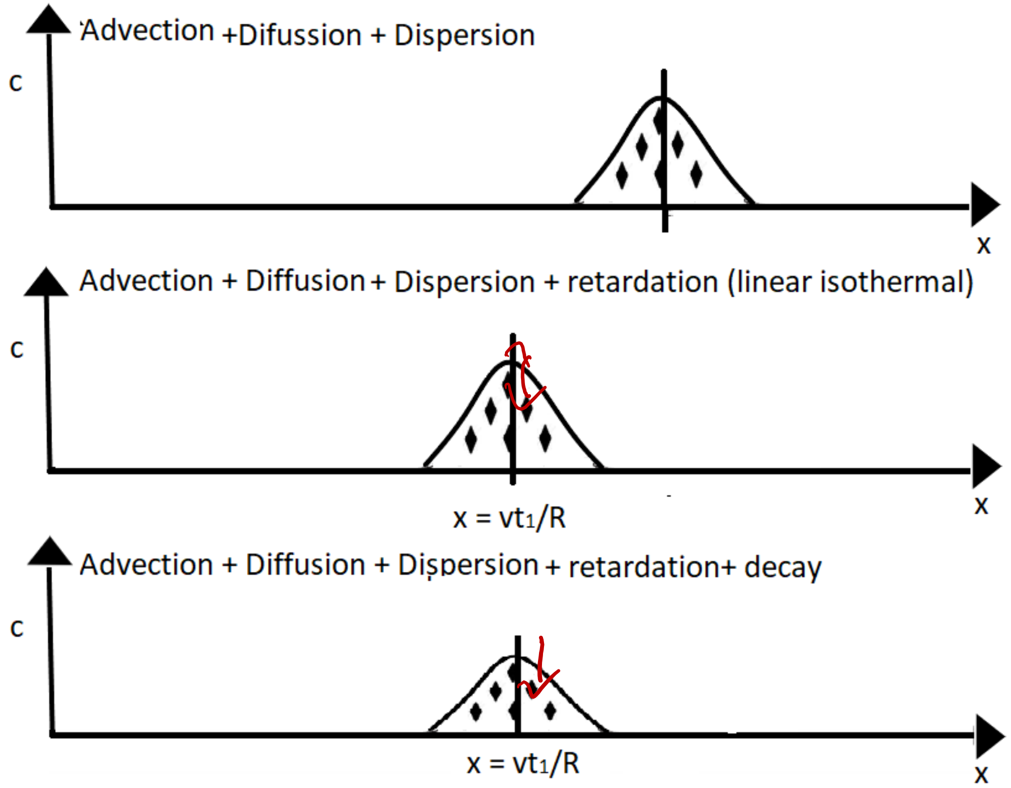
$$\frac{\partial C_{dec.}}{\partial t} = -\lambda\ C_0\ e^{-\lambda t} = -\lambda\ C$$

Effects of processes on a plume



Effects of processes on a plume – ctd.





Conservation of solute mass

- The temporal change in concentration is balanced by fluxes and sources/sink

$$\boxed{\frac{\partial C}{\partial t}} = \boxed{-}\nabla\left(\underline{\vec{j}_{dif}} + \underline{\vec{j}_{dis}} + \underline{\vec{j}_{adv}}\right) + Q$$

source/sink

- Including retardation

$$\boxed{R}\frac{\partial C}{\partial t} = \nabla\left(\underbrace{D + \mathbf{D}_s}_{D_H}\right)\nabla C - \nabla\left(\vec{q}C\right) + \underline{Q\boxed{'}}$$

Relevant parameters

- Permeability
- Hydraulic conductivity
- (effective) Porosity
- Transmissivity
- Storativity
- Density
- Viscosity
- Compressibility

- Diffusion/Dispersion coeff.
- *Transport velocity*
- Distribution coeff.
- Retardation factor
- Decay constant

Lessons learned

- Concept of continuum mechanics
- Spatial scale and relevant processes define the mathematical model
- Definition of a representative elementary volume
- The equation for horizontal groundwater flow and the relevant parameters
- Transport processes of solute mass in groundwater
- Relevant parameters